Interlaminar Stresses around Circular Cutouts in Composite Plates under Tension

S. Tang*
Grumman Aerospace Corporation, Bethpage, N. Y.

A boundary-layer thoery for isotropic elastic plates with a circular cutout developed by Reiss, is extended to laminated composites. An analytical solution is obtained for the extension of an infinite plate with a circular hole. The interlaminar shear stresses and the normal or "peel" stress near and at the edge of the hole are estimated for orthotropic plates. Numerical examples are given for (0 deg/90 deg)$_s$ and ($\pm$45 deg)$_s$ laminates. Results are compared with available finite-element solutions.

I. Introduction

DUE to the importance of design applications, continuous fiber reinforced resin matrix laminated composites containing cutouts have been the subject of recent studies.[1-5] From these results, the static failure strengths are obtained with good quantitative prediction compared to experiments. Delamination at free edges is another failure mode of laminated composites, especially under fatigue loadings. Extensive delamination at free edges is reported on static strength as well as fatigue strength, with and without notches.[6-10] Hence, the determination of interlaminar shear stresses and the normal or "peel" stress is a current research topic. However, all the previous studies are restricted to straight boundaries[11-27] except for finite element solutions in Refs. 14 and 28-30. There is no analytical closed-form solution available to predict interlaminar stresses for curvilinear boundaries. Therefore, the present paper attempts to study the interlaminar stresses at a circular cutout in an "infinite" composite laminate under uniform tensile load. The boundary-layer theory for composite laminates in Ref. 20 is extended here to the formulation of polar coordinates, and the title problem is investigated for the case of orthotropic composite plates.

The laminate considered here is under inplane loading symmetric about the midplane. The construction of the laminate is also midplane symmetric so that there is no stretching/bending coupling of the plate due to external load. The force resultants around the hole of an orthotropic or anisotropic plate are given by Refs. 31 and 32. The radial and shear force resultants at the edge of the hole are zero. From the plane stress solution, the strains at the edge of the hole can be calculated by the force resultant-strain relation. Due to the compatibility of deformation, the strains of individual layers are the same as the laminate. Therefore, the stresses of each layer may then be computed from the layer stress-strain law. However, the computed radial and shear stresses of each layer along the contour of the hole are in general not zero. Because there exists a three-dimensional state of stresses at the free edge of each layer where the plane stress solution cannot predict.

The region of the plate adjacent to and including the edge, where the plane stress solution may not be adequate, is called the boundary layer. To obtain the governing equations in this region, the boundary layer is "stretched" and the stresses are expanded in asymptotic power series about a thickness to radius parameter.[33] The lowest order approximation of the equilibrium equations obtained here are the same boundary-layer equations (in polar form) used by Pipes and Pagano.[12,19] Their approach is based on a displacement formulation. The present approach is based on a stress formulation. Therefore, the conditions of compatibility in terms of stresses are also obtained in the form of the lowest order approximation. The advantage of the present approach is that the boundary-layer stresses may be separated into two groups. Two components of stresses can be analyzed by a modified torsion formulation and the remaining stresses can be analyzed by a modified plane strain formulation. For a straight edge problem results obtained by using the present approach[21] are in good agreement with results by Pipes and Pagano.[12]

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*Senior Structural Mechanics Engineer, Advanced Composites Group.

II. Formulation

To analyze the title problem, the cylindrical coordinate system r, θ, Z is introduced. An infinite plate of total thickness $t=2H$ is made up by $2m$ layers of orthotropic lamina of thickness $2h$ (i.e., $2H=4mh$). The plate has a circular hole of radius R. The origin of the coordinate system is fixed at the center of the hole on the midplane of the plate, $Z=0$.

The laminate is stretched by a uniform tensile stress p along x axis at far field. Both the top and the bottom faces of the plate as well as the edge of the hole are free of traction. Then each layer, with its boundary $r \geq R$ and $Z=\pm h$, can be treated as a three-dimensional elastic body. The following dimensionless variables are introduced

$$\xi=(r-R)/R \qquad \xi \geq 0$$

$$\rho=Z/h \qquad |\rho| \leq \rho_H \qquad \rho_H=H/h$$

$$\epsilon=h/R$$

so that the plate boundaries are $\rho=\pm H/h$ and $\xi=0$. The equilibrium equations in terms of stresses can be shown to be[33]

$$\tau_{rz,\rho}+\epsilon\left[\sigma_{r,\xi}+\frac{1}{1+\xi}(\tau_{r\theta,\theta}+\sigma_r-\sigma_\theta)\right]=0$$

$$\tau_{z\theta,\rho}+\epsilon\left[\tau_{r\theta,\xi}+\frac{1}{1+\xi}(\sigma_{\theta,\theta}+\tau_{r\theta})\right]=0$$

$$\sigma_{z,\rho}+\epsilon\left[\tau_{rz,\xi}+\frac{1}{1+\xi}(\tau_{z\theta,\theta}+\tau_{rz})\right]=0 \qquad (1)$$

where body forces are not included. The stress components in Eq. (1) may be represented by asymptotic power series expansion in ϵ. The lowest order of this expansion of Eq. (1) is the same as the classical plane stress equations in polar coordinates.[33] This derivation will not be repeated here.

III. The Boundary Layer

To obtain the governing equations in the boundary-layer region, the "stretched" boundary-layer variable[33] η is introduced as

$$\eta = \xi / \epsilon \tag{2}$$

the boundary-layer stresses symbolized by $f(\eta,\theta,\rho;\epsilon)$ may be expanded in asymptotic power series of ϵ

$$f(\eta,\theta,\rho;\epsilon) = \sum_{n=0}^{\infty} f^n(\eta,\theta,\rho)\epsilon^n \tag{3}$$

The f^n are called the boundary-layer stress coefficients, and $f^n = 0$ when $n<0$. Introducing Eqs. (2) and (3) into the equilibrium and the compatability equations of stresses and collecting coefficients of equal power of ϵ yields the equilibrium equations of the boundary layer stresses

$$f^n_{zr,\rho} + f^n_{r,\eta} + \eta(f^{n-1}_{zr,\rho} + f^{n-1}_{r,\eta}) + f^{n-1}_{r\theta,\theta} + f^{n-1}_r - f^{n-1}_\theta = 0$$

$$f^n_{z,\rho} + f^n_{zr,\eta} + \eta(f^{n-1}_{z,\rho} + f^{n-1}_{zr,\eta}) + f^{n-1}_{z\theta,\theta} + f^{n-1}_{zr} = 0$$

$$f^n_{z\theta,\rho} + f^n_{r\theta,\eta} + \eta(f^{n-1}_{z\theta,\rho} + f^{n-1}_{r\theta,\eta}) + f^{n-1}_{\theta,\theta} + 2f^{n-1}_{r\theta} = 0 \tag{4}$$

When $n=0$, the equations of equilibrium are simplified to

$$f^0_{zr,\rho} + f^0_{r,\eta} = 0 \tag{5a}$$

$$f^0_{z,\rho} + f^0_{zr,\eta} = 0 \tag{5b}$$

$$f^0_{z\theta,\rho} + f^0_{r\theta,\eta} = 0 \tag{5c}$$

These are the same boundary-layer equations (in polar form) used by Pipes and Pagano.[12] For no loss of generality, the corresponding compatibility equations ($n=0$) at $\theta = \pi/2$ are given as

$$\left[(S_{12} + \frac{S_{66}}{2})f^0_r + S_{11}f^0_\theta + S_{13}f^0_z - S_{16}f^0_{r\theta}\right]_{,\eta\eta} - \frac{S_{66}}{2}f^0_{z,\rho\rho} = 0 \tag{6a}$$

$$\left[(S_{23} + \frac{S_{44}}{2})f^0_r + S_{13}f^0_\theta + S_{33}f^0_z - (S_{45} + S_{36})f^0_{r\theta}\right]_{,\eta\eta} + \left[S_{22}f^0_r + S_{12}f^0_\theta + (S_{23} + \frac{S_{44}}{2})f^0_z - S_{26}f^0_{r\theta}\right]_{,\rho\rho} = 0 \tag{6b}$$

$$\frac{S_{55}}{2}f^0_{r,\eta\eta} - \left[S_{12}f^0_r + S_{11}f^0_\theta + (S_{13} + \frac{S_{55}}{2})f^0_z - S_{16}f^0_{r\theta}\right]_{,\rho\rho} = 0 \tag{6c}$$

$$\left[(S_{12} + \frac{S_{66}}{2})f^0_r + S_{11}f^0_\theta + (S_{13} + \frac{S_{55}}{2})f^0_z - S_{16}f^0_{r\theta}\right]_{,\eta\rho} + \frac{S_{55}}{2}f^0_{zr,\eta\eta} + \frac{S_{66}}{2}f^0_{zr,\rho\rho} = 0 \tag{6d}$$

$$[S_{26}f^0_r + S_{16}f^0_\theta + S_{36}f^0_z]_{,\eta\rho} + S_{55}f^0_{z\theta,\eta\eta} + S_{66}f^0_{z\theta,\rho\rho} - S_{45}f^0_{zr,\eta\eta} = 0 \tag{6e}$$

$$(S_{26}f^0_r + S_{16}f^0_\theta + S_{36}f^0_z - S_{66}f^0_{r\theta})_{,\rho\rho} - S_{45}f^0_{zr,\eta\rho} = 0 \tag{6f}$$

where the S_{ij}'s are the anisotropic compliance constants of the strain-stress relation $[\epsilon_{ik}] = [S_{ij}][\sigma_{jk}]$.

Now the boundary-layer region can be identified as two problems,[33] namely, a modified torsion problem T which involves only $f^0_{r\theta}$ and $f^0_{z\theta}$ in Eq. (5c) and a modified-plane strain problem P which involves f^0_r, f^0_z, and f^0_{zr} in Eqs. (5a) and (5b). For the T problem, a function Ψ is introduced such that

$$f^0_{r\theta} = \Psi_{,\eta} \qquad f^0_{z\theta} = \Psi_{,\rho} \tag{7a}$$

and Eq. (5c) becomes

$$\nabla^2 \Psi = 0 \tag{7b}$$

where $\nabla^2 = \partial^2/\partial\eta^2 + \partial^2/\partial\rho^2$. The boundary conditions for this problem are

$$\lim_{\eta\to\infty} \Psi_{,\rho} = 0 \tag{7c}$$

$$\Psi_{,\rho}(\rho = \pm\rho_H) = 0 \tag{7d}$$

$$[\Psi_{,\rho}(\rho = \pm 1)]_i = [\Psi_{,\rho}(\rho = \mp 1)]_{i\mp 1} \tag{7e}$$

$$\Psi_{,\eta} + \tau^0_{r\theta} = 0, \qquad \eta = 0 \tag{7f}$$

Equation (7c) indicates that the boundary-layer stress coefficient $f^0_{z\theta}$ vanishes as $\eta\to\infty$. Equations (7d) and (7e) require $f^0_{z\theta}$ to vanish at top and bottom faces of the plate and/or to be continuous at interfaces. Equation (7f) is the condition for the matching of the shear stresses at $\eta = 0$.

For the P problem, a stress function ϕ is introduced so that

$$f^0_r = \phi_{,\rho\rho} \qquad f^0_{zr} = -\phi_{,\rho\eta} \qquad f^0_z = \phi_{,\eta\eta} \tag{8}$$

and the remaining two equations (5a) and (5b) are automatically satisfied. Function f^0_θ may be obtained from the strain displacement and strain-stress relationship by power series expansion similar to Eq. (3)

$$f^0_\theta = -(S_{12}f^0_r + S_{13}f^0_z - S_{16}f^0_{r\theta})/S_{11} \tag{9}$$

It can be shown that the boundary-layer strain component ϵ^0_θ is zero. Upon substitution of Eqs. (7a), (8), and (9) into the compatibility conditions (6a)-(6d), one finds that Eqs. (6a), (6c), and (6d) are satisfied identically and (6b) leads to

$$M_a\phi_{,\eta\eta\eta\eta} + M_b\phi_{,\eta\eta\rho\rho} + M_c\phi_{,\rho\rho\rho\rho} = -M_d\Psi_{,\eta\rho\rho} - M_e\Psi_{,\eta\eta\eta} \tag{10}$$

with

$$M_a = S_{33} - \frac{S^2_{13}}{S_{11}} \qquad M_b = 2(S_{23} + \frac{S_{44}}{2} - \frac{S_{13}S_{12}}{S_{11}})$$

$$M_c = S_{22} - \frac{S^2_{12}}{S_{11}} \qquad M_d = \frac{S_{12}S_{16}}{S_{11}} - S_{26}$$

$$M_e = \frac{S_{13}S_{16}}{S_{11}} - S_{36} - S_{45}$$

Equation (10) reduces to $\nabla^4\phi = 0$ for an isotropic elastic body. The corresponding boundary conditions for the P problem are

$$\lim_{\eta\to\infty} \{\phi_{,\rho\rho};\ \phi_{,\rho\eta}\} = 0 \tag{11a}$$

$$\phi_{,\eta\eta}(\rho = \pm\rho_H) = 0 \tag{11b}$$

$$\phi_{,\eta\rho}(\rho = \pm\rho_H) = 0 \tag{11c}$$

$$\phi_{,\rho\rho} + \sigma^0_r = 0 \qquad \eta = 0 \tag{11d}$$

$$\phi,_{\rho\eta}=0 \qquad \eta=0 \tag{11e}$$

$$[\phi,_{\eta\eta}(\rho=+1)]_i-[\phi,_{\eta\eta}(\rho=\pm 1)]_{i+1}=0 \tag{11f}$$

IV. Plane Stress Solution

Considering an anisotropic plate with a circular or elliptic hole subjected to a uniform tensile stress p at far field, one finds that the equations of equilibrium of plane stress are satisfied by the introduction of the Airy function.[34]

$$\sigma_x=\frac{\partial^2 U}{\partial x^2} \qquad \sigma_y=\frac{\partial^2 U}{\partial_y{}^2} \qquad \tau_{xy}=-\frac{\partial^2 U}{\partial_x\partial_y}$$

The inplane compatibility condition for an anisotropic medium may be expressed in the following form[30]

$$b_{22}\frac{\partial^4 U}{\partial x^4}-2b_{26}\frac{\partial^4 U}{\partial x^3\partial y}+(2b_{12}+b_{66})\frac{\partial^4 U}{\partial x^2\partial y^2}$$

$$-2b_{16}\frac{\partial^4 U}{\partial x\partial y^3}+b_{11}\frac{\partial^4 U}{\partial y^4}=0$$

which is a generalization of the well-known biharmonic equation for an isotropic solid. Here the b_{ij}'s are the elements of the flexibility matrix and a_{ij}'s are the elements of the stiffness matrix.

$$\epsilon_i=b_{ij}\sigma_j, \quad \sigma_i=a_{ij}\epsilon_j, \quad [b_{ij}]=[a_{ij}]^{-1}$$

The solution of the above differential equation depends on the roots of the characteristic equation

$$b_{11}s^4-2b_{16}s^3+(2b_{12}+b_{66})s^2-2b_{26}s+b_{22}=0$$

On the basis of energy consideration, Lekhnitskii[37] proved that this equation cannot have real roots. Upon denoting the roots by s_1, s_2, s_3, and s_4,

$$s_1=\alpha_1+i\beta_1, \quad s_2=\alpha_2+i\beta_2, \quad s_3=\alpha_1-i\beta_1, \quad s_4=\alpha_2-i\beta_2$$

where α_1, α_2, β_1, β_2 are real constants.

The laminate stress components for infinite, elastic, anisotropic plates with elliptical or circular holes, subjected to uniform tension p at far field, are given by the following formulas[31,32]

$$\sigma_x=p\cos^2\alpha+2Re[s_1^2\varphi_0'(Z_1)+s_2^2\Psi_0'(Z_2)]$$

$$\sigma_y=p\sin^2\alpha+2Re[\varphi_0'(Z_1)+\Psi_0'(Z_2)] \tag{12}$$

$$\tau_{xy}=p\sin\alpha\cos\alpha-2Re[s_1\varphi_0'(z_1)+s_2\Psi_0'(Z_2)]$$

where

$$\varphi_0'(Z_1)=\frac{-ip\pi}{4(s_1-s_2)(a+is_1b)}[1-Z_1(Z_1^2-a^2-b^2s_1^2)^{-1/2}]$$

$$\pi_1=b(s_2\sin 2\alpha+2\cos^2\alpha)+i\,a(2s_2\sin^2\alpha+\sin 2\alpha)$$

$$\Psi_0'(Z_2)=\frac{ip\pi_2}{4(s_1-s_2)(a+is_2b)}[1-Z_2(Z_2^2-a^2-b^2s_2^2)^{-1/2}]$$

$$\pi_2=b(S_1\sin 2\alpha+2\cos^2\alpha)+i\,a(2S_1\sin^2\alpha+\sin 2\alpha)$$

$$Z_1=x+s_1y \qquad Z_2=x+s_2y$$

α is the angle between loading direction and $0x$ axis, a,b are the major and minor axes of ellipse, and the stresses at each individual layer may be found from the following constitutive equations for each layer

$$\begin{bmatrix}\sigma_x^0\\ \sigma_y^0\\ \tau_{xy}^0\end{bmatrix}^k=\begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ Q_{12} & Q_{22} & Q_{26}\\ Q_{16} & Q_{26} & Q_{66}\end{bmatrix}^k\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}^k \tag{13}$$

Here, σ^0 indicates the layer stress, k identifies the layer number, k will be dropped in the subsequent derivation for simplicity. Since the overall strains are the same as the strains at each individual layer, the values of ϵ_x, ϵ_y, and γ_{xy} can be computed from the overall stresses in Eq. (12) through the strain-stress relationship $\epsilon_i=b_{ij}\,\sigma_j$.

There is an approximate expression in the form of a polynomial for the stress component σ_x when the load is applied along the $0x$ axis direction.[35] The expression is restricted to a circular hole for an orthotropic plate. Furthermore, the other two stress components σ_y and τ_{xy} are not available in Ref. 35, and individual layer stresses cannot be computed without the complete information of the state of stresses of the overall laminate. Therefore, stresses in Eq. (12) are calculated by digital computers.

V. Analysis of the Boundary-Layer Near Hole

For the T problem, the stress function ψ is chosen to be

$$\psi=C\exp(-\beta\pi\eta)\cos(\beta\pi\rho) \qquad \beta>0 \tag{14}$$

so that it automatically satisfies the differential equation (7b). The boundary-layer stress coefficients are given as

$$f_{r\theta}^0=-C\beta\pi\exp(-\beta\pi\eta)\cos(\beta\pi\rho) \tag{15}$$

$$f_{z\theta}^0=-C\beta\pi\exp(-\beta\pi\eta)\sin(\beta\pi\rho) \tag{16}$$

To determine the constant β, the boundary condition (7d) is used, i.e., $f_{z\theta}^0=0$ on $\rho=\pm\rho_H$. This implies $\rho_H=\pm 2m$ and $\beta=\frac{1}{2}m$ ($2m$ is the number of layers). Since β is a positive number, the boundary condition (7c) is satisfied, i.e., $f_{z\theta}^0=0$ as $\eta\to\infty$. The constant C may be determined by boundary condition (7f). It is computed on the basis of matching the total inplane shear flow across the thickness of each layer, whereupon

$$C=\frac{(Q_{16}b_{11}+Q_{26}b_{12}+Q_{66}b_{16})(\rho_{i+1}-\rho_i)\sigma_x(x=0,y=R)}{\sin(\frac{\pi}{2m}\rho_{i+1})-\sin(\frac{\pi}{2m}\rho_i)} \tag{17}$$

It may be observed that if the plate construction is made of bidirectional laminate (0 deg/90 deg)$_s$ and the applied load at far field is along $\theta=0$ deg direction, then $Q_{16}=0$, $Q_{26}=0$, $b_{16}=0$, and hence $C=0$. Therefore, in this case, $f_{r\theta}^0=0$ and $f_{z\theta}^0=0$ across the thickness of the whole laminate at $\theta=\pi/2$.

For the P problem, the differential equation takes the following form for the present problem

$$M_a\phi,_{\eta\eta\eta\eta}+M_b\phi,_{\zeta\zeta\eta\eta}+M_c\phi,_{\zeta\zeta\zeta\zeta}$$

$$=-(M_d-M_e)C\frac{\pi^3}{8m^3}\exp(-\frac{\pi}{2m}\eta)\cos(\frac{\pi}{2m}\rho) \tag{18}$$

The solution of Eq. (18) may be obtained as

$$\phi=\Big[A\cdot\exp(-\lambda_1\eta)+B\cdot\exp(-\lambda_2\eta)+\bar{K}\cdot\exp(-\frac{\pi}{2m}\eta)\Big]\cos(\frac{\pi}{2m}\rho)$$

with

$$\bar{K}=\frac{-2mC(M_d-M_c)}{\pi(M_a-M_b+M_c)} \qquad \lambda_{1,2}=\frac{\pi}{2m}\left[\frac{M_b\pm\sqrt{M_b^2-4M_aM_c}}{2M_a}\right]^{1/2}$$

where the plus and minus sign correspond to the subscript 1 and 2 respectively. By definition, the boundary-layer stress coefficients are given as follows

$$f^0_r = -\frac{\pi^2}{4m^2}\left[A\cdot\exp(-\lambda_1\eta)+B\cdot\exp(-\lambda_2\eta)\right.$$

$$\left.+\bar{K}\cdot\exp(-\frac{\pi\rho}{2m})\right]\cdot\cos(\frac{\pi\rho}{2m}) \tag{19}$$

$$f^0_{zr} = -\frac{\pi}{2m}\left[A\lambda_1\exp(-\lambda_1\eta)+B\lambda_2\exp(-\lambda_2\eta)\right.$$

$$\left.+\bar{K}\frac{\pi}{2m}\exp(-\frac{\pi\eta}{2m})\right]\cdot\sin(\frac{\pi\rho}{2m})$$

$$f^0_z + \left[A\lambda_1^2\exp(-\lambda_1\eta)+B\lambda_2^2\exp(-\lambda_2\eta)\right.$$

$$\left.+\frac{\pi^2}{4m^2}K\exp(-\frac{\pi\eta}{2m})\right]\cdot\cos(\frac{\pi\rho}{2m})$$

$$f^0_\theta = -(S_{12}f^0_r+S_{13}f^0_r+S_{16}f^0_{r\theta})/S_{11}$$

and the total stress components are obtained by combining the stresses from the interior domain and the corresponding stress coefficients of the boundary layer

$$(\sigma_r)_t = \sigma^0_r + f^0_r$$

$$(\sigma_\theta)_t = \sigma^0_\theta + f^0_\theta$$

$$(\tau_{r\theta})_t = \tau^0_{r\theta} + f^0_{r\theta}$$

$$(\tau_{zr})_t = f^0_{zr}$$

$$(\tau_{z\theta})_t = f^0_{z\theta}$$

$$(\sigma_z)_t = f^0_z \tag{20}$$

It may be observed that as $\eta\to\infty$, $f^0_r=f^0_{zr}=f^0_z=f^0_\theta=0$; hence, the boundary condition (11a) is satisfied, and also condition (11c) is satisfied by $f^0_{zr}=0$ on $\rho=\pm\rho_H$. The two constants A and B are determined by the two boundary conditions (11d) and (11e), on the basis of match total force across the thickness of each layer.

$$A=(q\lambda_2+\frac{\pi\bar{K}}{2m})/(\lambda_2-\lambda_1)$$

$$B=-(\frac{\pi\bar{K}}{2m}+q\lambda_1)/(\lambda_2-\lambda_1)$$

with

$$q=\frac{2m\sigma_x(\rho_{i+1}-\rho_i)(Q_{12}b_{11}+Q_{22}b_{12}+Q_{26}b_{16})}{\pi\sin\left[(\frac{\pi\rho i+1}{2m})-\sin(\frac{\pi\rho i}{2m})\right]}-\bar{K}$$

However, the condition $f^0_z=0$ on $\rho=\pm\rho_H$ cannot be satisfied by the assumed stress function. Hsu and Herakovich encountered similar difficulty in their displacement functions solution.[27]

By similar power series expansion as Eq. (3), the displacement components may be obtained through strain displacement and strain-stress relationships.

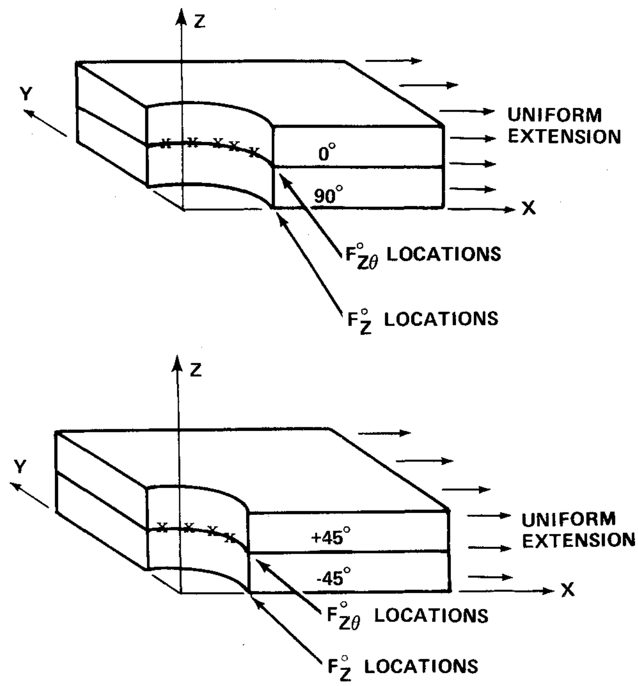

Fig. 1 Laminate constructions.

$$v^1=R\int\left[(S_{26}-\frac{S_{16}^2}{S_{11}})f^0_r+(S_{36}-\frac{S_{16}S_{13}}{S_{11}})f^0_z\right.$$

$$\left.+(S_{66}-\frac{S_{16}^2}{S_{11}})\cdot f^0_{r\theta}\right]\mathrm{d}\eta \tag{21}$$

$$w^1=R\int\left[(S_{23}-\frac{S_{13}S_{12}}{S_{11}})f^0_r+(S_{33}-\frac{S_{13}^2}{S_{11}})f^0_z\right.$$

$$\left.+(S_{36}-\frac{S_{13}S_{16}}{S_{11}})\cdot f^0_{r\theta}\right]\mathrm{d}\zeta$$

VI. Numerical Results and Discussion

Sample problems are worked for laminates of two different constructions (0 deg/90 deg)$_s$ and ($\pm$45 deg)$_s$ as shown in Fig. 1. For comparison purposes (with Ref. 14), the (0 deg/90 deg)$_s$ laminate is made of boron/epoxy with the following typical properties:

$$E_{11}=30.7\times10^6 \text{ psi} \qquad E_{22}=E_{33}=2.89\times10^6 \text{ psi}$$

$$G_{12}=G_{13}=G_{23}=0.723\times10^6 \text{ psi}$$

$$\nu_{12}=\nu_{13}=\nu_{23}=0.21$$

and the properties of the other laminate are calculated by using material properties of typical graphite/epoxy

$$E_{11}=20.0\times10^6 \text{ psi} \qquad E_{22}=E_{33}=2.1\times10^6 \text{ psi}$$

$$G_{12}=G_{13}=G_{23}=0.85\times10^6 \text{ psi}$$

$$\nu_{12}=\nu_{13}=\nu_{23}=0.21$$

For the [0 deg/90 deg]$_s$ laminate containing a circular hole (Fig. 2), the interlaminar shear stress $f^0_{z\theta}$ is zero along $\theta=0$ deg and $\theta=90$ deg as expected becasue there is no inplane shear stress $\tau^0_{r\theta}$ at these two locations, and $f^0_{z\theta}$ is the direct product due to the matching of the inplane shear stresses at the free edge. However, away from these two points, inplane shear stress is developed due to the curvilinear boundary of the hole, and consequently $f^0_{z\theta}$ grows rapidly away from these

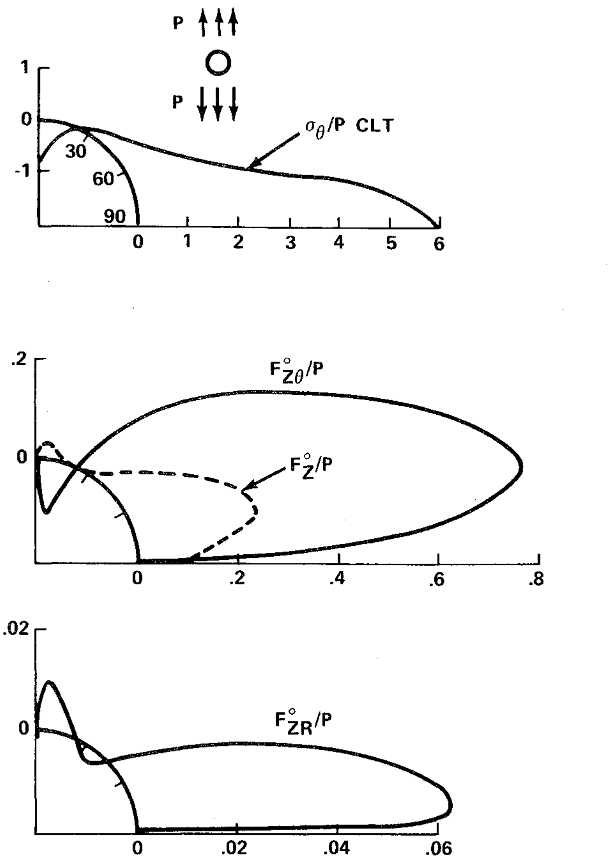

Fig. 2 CLT and interlaminar stresses around circular hole, *B/E*, [0 deg/90 deg]$_s$.

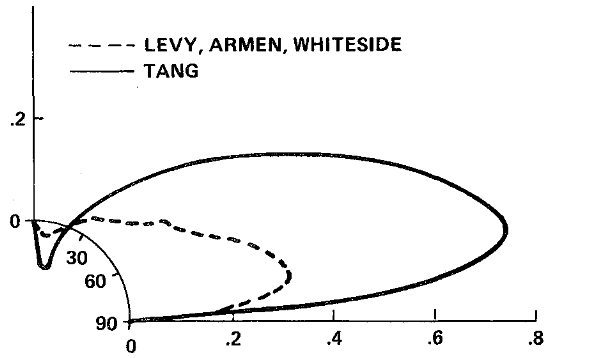

Fig. 3 *B/E* [0 deg/90 deg]$_s$ interlaminar shear-stress distribution, $f^0_{z\theta}$ comparison with Ref. 14.

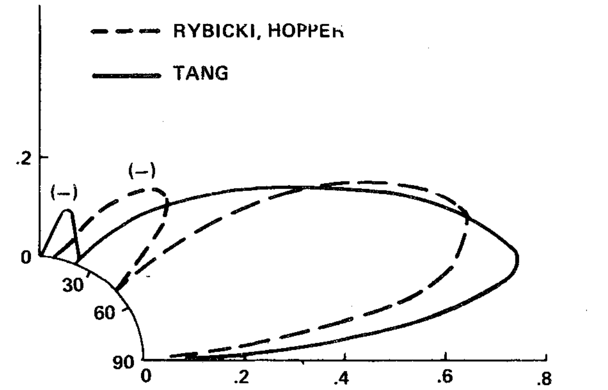

Fig. 4 *B/E* [0 deg/90 deg]$_s$ interlaminar shear-stress distribution, $f^0_{z\theta}$ comparison with Ref. 28.

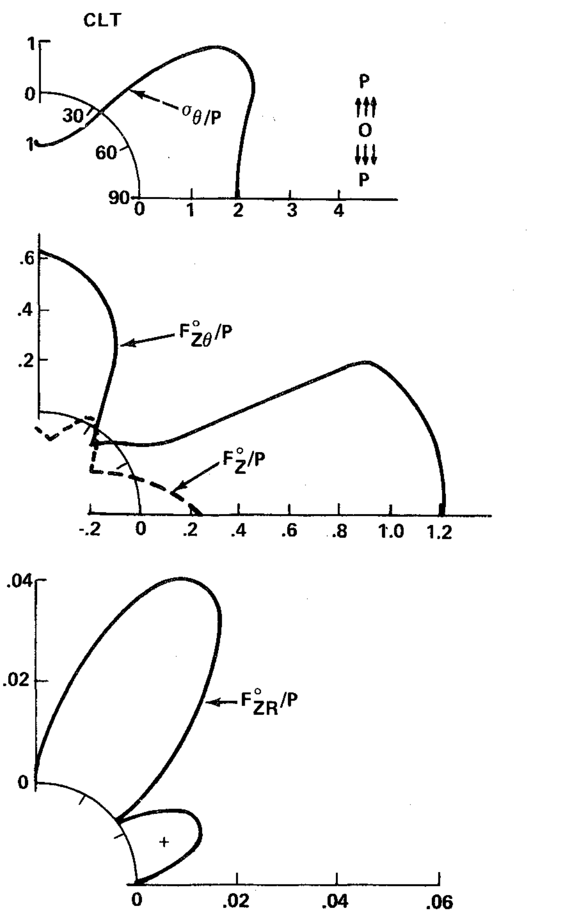

Fig. 5 CLT and interlaminar stresses around circular hole, *GR/E*, (±45 deg)$_s$.

points. The dimensionless laminate tangential stress (CLT) around the hole is in excellent agreement with the finite element of Levy et al.[14] as expected. In computing the interlaminar shear stresses, they used a model generated by Puppo and Evensen.[11] This model consisted of alternating layers of orthotropic membranes and interlaminar shear elements. It should be noted that in a straight edge problem, the maximum value of interlaminar shear stress computed in Ref. 11 is about half of the value obtained by using the three-dimensional boundary-layer equations such as Pipes and Pagano[12] and the present approach.[21] Therefore, it is not surprising to find the $f^0_{z\theta}$ distribution around the hole calculated from the present analysis here is twice the size of that obtained by Levy et al.[14] Furthermore, it is interesting to observe that the shapes of the stress distributions are almost identical (Fig. 3).

Rybicki and Hopper[28] and Rybicki and Schumueser[36] present three-dimensional finite-element solutions of composite laminates containing a circular hole. The comparison of the interlaminar shear-stress component $f^0_{z\theta}$ from the present analysis with that of Rybicki and Hopper is shown in Fig. 4. It should be pointed out that the material properties they use were somewhat different

$$E_l = 30\times 10^6 \text{ psi} \qquad E_2 = 3\times 10^6 \text{ psi}$$

$$G_{12} = 1\times 10^6 \text{ psi and } \nu_{12} = 0.336$$

From Fig. 4, it can be seen that the maximum values of $f^0_{z\theta}$ from these two results are different by 10%. Also the peaks and valleys of the stress distributions are not quite lined up at the same locations. The discrepancy between these two results may be due to the difference of the material properties used, especially G_{12} and ν_{12}.

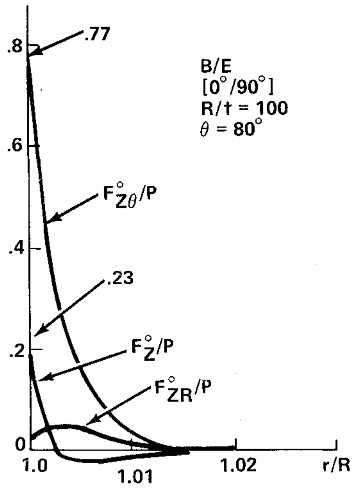

Fig. 6 Interlaminar stresses die-away from edge of circular hole, *B/E*, [0 deg/90 deg]$_s$, $R/t = 100$.

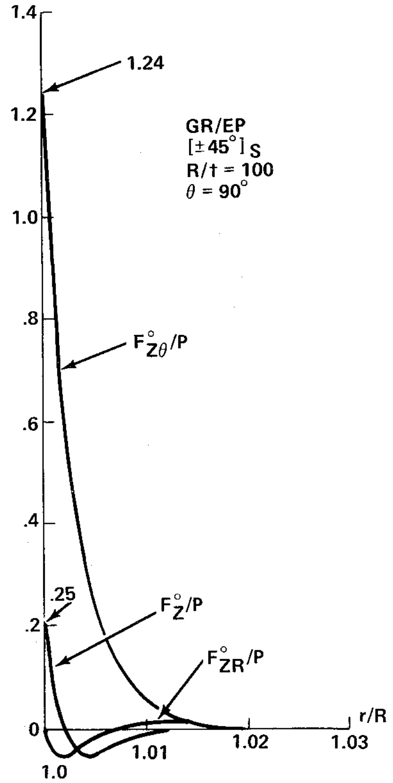

Fig. 7 Interlaminar stresses die-away from edge of circular hole, *GR/E*, (±45 deg)$_s$, $R/t = 100$.

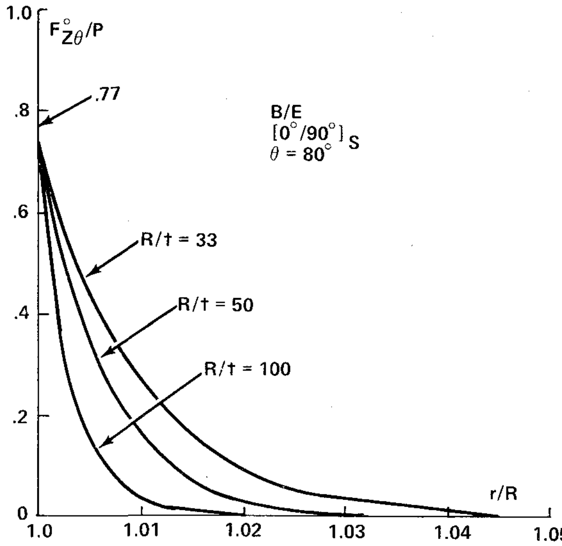

Fig. 8 Interlaminar shear-stress die-away from hole edge as function of R/t ratio.

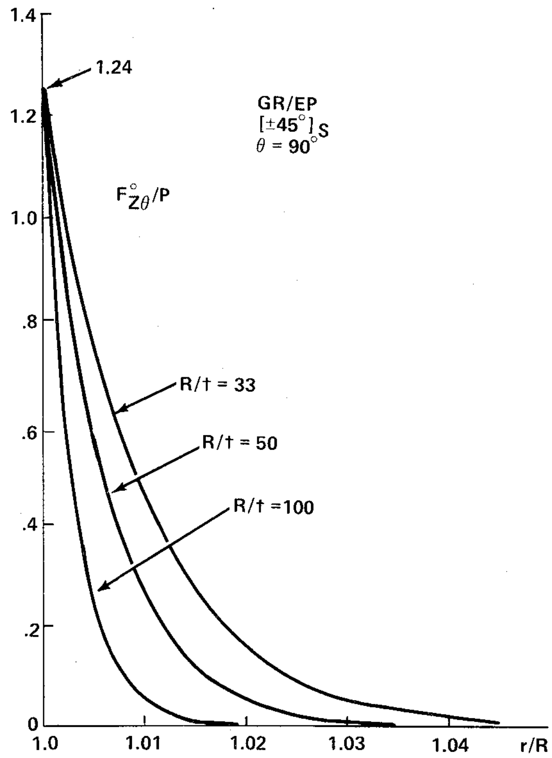

Fig. 9 Interlaminar shear stress die-away from hole edge as function of R/t ratio.

For the [±45 deg]$_s$ laminate containing a circular hole, the magnitude of $f^0_{z\theta}$ has high values at $\theta = 0$ deg and $\theta = 90$ deg because there are high inplane shear stresses $\tau^0_{r\theta}$ at these locations (Fig. 5), and this situation is just the opposite of the [0 deg/90 deg]$_s$ laminate. The value of $f^0_{z\theta}/p$ at $\theta = 90$ deg is 1.24 for the [±45 deg]$_s$ laminate. The corresponding value of f^0_{zx}/p for the straight-edge problem[21] is 0.59. There is approximately a factor of two between these two solutions. Therefore a straight-edge solution cannot be used as substitute for a circular cutout solution, even when the hole size is relatively large.

V. Conclusions

In general, from the present analysis, it is found that the circumferential component of interlaminar shear stress $f^0_{z\theta}$ is dominant (Figs. 2, 5-7). This component of interlaminar shear stress is caused by the balancing of the inplane shear stresses $\tau^0_{r\theta}$ between adjacent layers at the free edge. The other two components of interlaminar stresses f^0_{zr} and f^0_z are comparatively small. These findings agree with Levy et al.[14] and Rybicki and Schmueser.[36]

It is also found from the present analysis that the magnitude of the interlaminar stresses, around a circular hole, depends on the material and the construction of the laminate. Also the boundary-layer region is directly in proportion to the R/t ratio, the radius of hole-to-plate thickness ratio. For a large R/t ratio or a thin plate, the boundary effect dies away quickly from the edge of the hole. For a small R/t ratio or a thick plate, the boundary-layer effect dies away slowly (Figs. 8 and 9).

The present solution is valid for [0 deg/90 deg]$_s$ and [±45 deg]$_s$ laminates, and it cannot account for laminates of general construction. The difficulties arise from the matching of boundary conditions at the interfaces of the layers. Further studies are needed for general laminates containing a circular hole.

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